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# COMMUTATIVE TWISTED GROUP ALGEBRAS

BY

HARVEY A. SMITH

ABSTRACT. A twisted group algebra $L^1(A, G; T, \alpha)$ is commutative iff $A$ and $G$ are, $T$ is trivial and $\alpha$ is symmetric: $\alpha(\gamma, g) = \alpha(g, \gamma)$. The maximal ideal space $L^1(A, G; \alpha)\hat{}$ of a commutative twisted group algebra is a principal $G\hat{}$ bundle over $A\hat{}$. A class of principal $G\hat{}$ bundles over second countable locally compact $M$ is defined which is in 1-1 correspondence with the (isomorphism classes of) $C_\infty(M)$-valued commutative twisted group algebras on $G$. If $G$ is finite only locally trivial bundles can be such duals, but in general the duals need not be locally trivial.

**1. Introduction.** Twisted group algebras, as defined in [4], generalize a number of types of Banach algebras studied previously under other names (see [4] for examples). Essentially the same algebras are studied by Leptin in [12], [13], [14] under the name "generalized $L^1$ algebras". In particular, they generalize the group algebras studied by G. P. Johnson in [9] and also those studied by L. Auslander and C. C. Moore in [1] and by C. M. Edwards and J. T. Lewis in [5] (these were also called twisted group algebras in the latter paper).

In [9] G. P. Johnson studied the Bochner integrable $A$-valued functions on $G$, where $A$ is a commutative Banach algebra and $G$ is a separable locally compact Abelian group. He showed that the maximal ideal space of the resulting Banach algebra, $L^1(A, G) = A \otimes_\gamma L^1(G)$, was the topological product of $A\hat{}$, the maximal ideal space of $A$, with $G\hat{}$, the dual group of $G$. In [1] and [5], on the other hand, the convolution multiplication in $L^1(G)$ is altered by inserting a cocycle $\alpha$ with values in the unit circle. One thus obtains a new algebra $L^1(G, \alpha)$ with multiplication

$$(f \cdot h)(g) = \int_G f(y)h(y^{-1}g)\alpha(y, y^{-1}g)\,dy.$$

If $L^1(G, \alpha)$ is commutative then it is readily seen that $G$ is Abelian and $\alpha$ is symmetric: $\alpha(x, y) = \alpha(y, x)$. It then follows from results of Kleppner [11] that $L^1(G, \alpha)$ is isomorphic to $L^1(G)$ and its maximal ideal space is thus homeomorphic to $G\hat{}$.

We show in §2 that if a twisted group algebra is commutative then it is

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$L^1(A, G; \alpha)$ the common generalization of the algebras of Johnson and those described above. (The cocycle takes values in the unitary double centralizers of $A$ rather than in the unit circle.) In §3 we show that the previous results on the maximal ideal spaces generalize in the following way: the maximal ideal space of a commutative twisted group algebra is a $G^\wedge$ principal fibre bundle over $A^\wedge$. Busby has shown [2] that the cross-section algebras of Borel homogeneous Banach*-algebraic bundles (see Fell [6]) are twisted group algebras. This gives us the following curious duality for the commutative case: a homogeneous Borel Banach*-algebraic bundle over $G$ with $A$ as the fibre over the identity has as its dual a principal bundle over $A^\wedge$ with $G^\wedge$ as fibre. All of these dual principal $G^\wedge$ bundles are induced by a mapping, produced by the cocycle defining twisted multiplication, of $A^\wedge$ into the base space of a standard principal $G^\wedge$ bundle.

In §4 we define *characteristic* principal $G^\wedge$ bundles over a second-countable locally-compact Hausdorff space $M$ and show that every such bundle is the dual of a twisted group algebra over $G$. These bundles need not be locally trivial in general but we are able to show they are locally trivial when $G$ is finite.

2. **Twisted group algebras.** Let $A$ be a separable Banach*-algebra with a two-sided approximate identity of unit norm, $U(A)$ the group of unitary double centralizers of $A$, $\mathrm{Aut}_1(A)$ the automorphisms of $A$ which, together with their inverses, have norm one and $\mathrm{Inn}(A)$ the subgroup of $\mathrm{Aut}_1(A)$ consisting of inner automorphisms by members of $U(A)$: $I_u(a) = uau^*$. Let $G$ be a second-countable locally compact group, let $T: G \longrightarrow \mathrm{Aut}_1(A)$ and $\alpha: G \times G \longrightarrow U(A)$ be Borel mappings such that the induced composition mapping $T: G \longrightarrow \mathrm{Aut}_1(A)/\mathrm{Inn}(A)$ is continuous and for all $x, y, z$ in $G$, $a$ in $A$

(1) $(T(x)\alpha(y, z))\alpha(x, yz) = \alpha(x, y)\alpha(xy, z)$,

(2) $(T(x)T(y)a)\alpha(x, y) = \alpha(x, y)(T(xy)a)$,

(3) $\alpha(e, x) = \alpha(y, e) = I$, $T(e) = I$.

($e$ is the identity of $G$ and we also write $T$ for the extension of $T$ to $A^-$ the double-centralizer algebra of $A$.) The Banach space $A \otimes_\gamma L^1(G)$ of all Bochner integrable $A$-valued functions on $G$ is a Banach*-algebra under the product

$$(f \cdot h)(x) = \int f(y)\,[T(y)h(y^{-1}x)]\alpha(y, y^{-1}x)\,dy$$

(where integration is with respect to a left Haar measure) and the involution $f^*(x) = \alpha(x, x^{-1})^*[T(x)f(x^{-1})^*]\Delta(x^{-1})$ where $\Delta$ is the modular function of $G$. (For details see [4].) Since $A$ has a two-sided approximate identity, the twisted group algebra operations can be extended to $A^-$, the double centralizer algebra of $A$, and $L^1(A, G; T, \alpha)$ can be considered to be embedded in $L^1(A^-, G; T, \alpha)$ when convenient. (See, for instance, [3].)

We now consider the consequences of the Banach*-algebra $L^1(A, G; T, \alpha)$ being commutative. For this purpose it suffices to consider functions $af$ where $a$

is in the algebra $A^{\sim}$ and $f$ is in $L^1(G)$. Since $A^{\sim}$ has an identity, 1, by considering the commutativity of $1 \cdot f$ and $af$ one finds immediately that $T$ is essentially trivial. Similarly, by considering $af$ and $bf$ one finds that $A$ must be commutative. The commutation of $1 \cdot f$ with $1 \cdot h$ yields

$$\int_G f(y)[h(y^{-1}x)\alpha(y, y^{-1}x) - h(xy^{-1})\alpha(xy^{-1}, y)\Delta(y^{-1})]\,dy = 0$$

for all $f$ and $h$ in $L^1(G)$. Thus we have $h(y^{-1}x)\alpha(y, y^{-1}x) = h(xy^{-1})\alpha(xy^{-1}, y)\Delta(y^{-1})$ which is easily seen to imply that $G$ is Abelian and $\alpha(x, y) = \alpha(y, x)$ almost everywhere. Conversely if all these properties hold, $L^1(A, G; T, \alpha)$ is commutative. We summarize these observations in the following remark, which generalizes Theorem 7 in I of [5].

**Remark.** $L^1(A, G; T, \alpha)$ is commutative if and only if $A$ and $G$ are commutative, $T$ is (essentially) trivial, and $\alpha$ is (essentially) symmetric: $\alpha(x, y) = \alpha(y, x)$ almost everywhere.

In the rest of this paper we will only be concerned with this case and we will write $L^1(A, G; \alpha)$.

**3. The maximal ideal space of $L^1(A, G; \alpha)$.** Let $m$ be a multiplicative linear functional on $A$. By results of B. E. Johnson [8], $m$ can be extended to such a functional on $A^{\sim}$. In particular $\alpha_m = m(\alpha)$ is then a cocycle taking values in the unit circle. Since $\alpha_m(x, y) = \alpha_m(y, x)$ we know by results of A. Kleppner [11] that $\alpha_m$ is trivial, i.e. there is some Borel function $\psi_m$ on $G$, taking values in the unit circle, such that $\alpha_m(x, y) = \psi_m(x)\psi_m(y)/\psi_m(xy)$. Moreover, $\psi_m$ is clearly unique to within multiplication by a character of $G$.

**Lemma 1.** *For each multiplicative functional $m$ on $A$ and each character $\chi$ in $G^{\wedge}$, $m \otimes \psi_m\chi$ is a multiplicative functional on $L^1(A, G; \alpha)$.*

**Proof.** The action of $\psi_m\chi$ on $L^1(G)$ is $\psi_m\chi(f) = \int_G f(g)\psi_m(g)\chi(g)\,dg$. Thus for $f, h$ in $L^1(A, G; \alpha)$

$$[m \otimes \psi_m\chi](f \cdot h) = \int_G \psi_m(g)\chi(g) m\left[\int_G f(y)h(y^{-1}g)\alpha(y, y^{-1}g)\,dy\right] dg$$

$$= \int_G \int_G \psi_m(g)\chi(g) f_m(y) h_m(y^{-1}g)\alpha_m(y, y^{-1}g)\,dy\,dg \quad (\text{where } f_m(g) = m(f(g)) \text{ etc.})$$

$$= \int_G \int_G \chi(g) f_m(y) h_m(y^{-1}g)\psi_m(y)\psi_m(y^{-1}g)\,dy\,dg$$

$$= \int_G \chi(y) f_m(y)\psi_m(y)\,dy \cdot \int_G \chi(y^{-1}g) h_m(y^{-1}g)\psi_m(y^{-1}g)\,dg$$

$$= [m \otimes \psi_m\chi](f) \cdot [m \otimes \psi_m\chi](h).$$

**Lemma 2.** *Every nontrivial multiplicative functional on $L^1(A, G; \alpha)$ is of the form $m \otimes \psi_m \chi$.*

**Proof.** Let $\mu$ be a nontrivial multiplicative linear functional on $L^1(A, G; \alpha)$. For every $a$ in $A$, define $[\pi(\mu)](a) = \mu(a \cdot f)/\mu(f)$ for any $f$ in $L^1(A, G; \alpha)$ such that $\mu(f) \neq 0$. (By nontriviality of $\mu$ there is such an $f$ and by multiplicativity of $\mu$ the definition is independent of the choice of $f$.) Since

$$\frac{\mu(a \cdot f)}{\mu(f)} \cdot \frac{\mu(b \cdot f)}{\mu(f)} = \frac{\mu(a \cdot b \cdot f)}{\mu(f)}$$

$\pi(\mu)$ is a multiplicative linear functional on $A$. Moreover it is clear that $\pi(m \otimes \psi_m \chi) = m$, so $\pi$ maps the nontrivial multiplicative linear functionals of $L^1(A, G; \alpha)$ onto those of $A$. We now write $m$ for $\pi(\mu)$.

The functional $\mu$ (extended if necessary to $L^1(A^-, G; \alpha)$) can be restricted to $L^1(G) = 1 \otimes L^1(G)$. Then, by definition of $m$, $\mu(a \otimes f) = m(a)\mu(1 \otimes f)$ so, by linearity and continuity, $\mu(k) = \mu(1 \otimes k_m)$ for any $k$ in $L^1(A, G; \alpha)$. Thus for any $f, h$ in $L^1(G)$

$$\mu((1 \otimes f) \cdot (1 \otimes h)) = \mu\left(1 \otimes \int f(g)h(g^{-1}\gamma)\alpha_m(g, g^{-1}\gamma)\,dg\right)$$

and we see that $\mu$ restricts to a multiplicative functional on $L^1(G)$ under the "twisted" convolution product defined by this last integral. By the results of Kleppner [11], $\alpha_m$ is the coboundary of a Borel function $\psi_m$ taking values in the unit circle: $\alpha_m(g, \gamma) = \psi_m(g)\psi_m(\gamma)/\psi_m(g\gamma)$ and $\psi_m$ is unique to within a character of $G$. Choosing a particular $\psi_m$ we see that division by $\psi_m$ produces an isomorphism of $L^1(G)$ with the twisted $\alpha_m$ multiplication to $L^1(G)$ with the usual convolution multiplication. Thus the action of $\mu$ is merely that of a character together with multiplication by $\psi_m$: $\mu(1 \otimes f) = \int_G \psi_m(g)\chi(g)f(g)\,dg$ and the action of $\mu$ on $A \otimes_\gamma L^1(G)$ is as asserted.

**Lemma 3.** *The mapping $\pi$: $L^1(A, G; \alpha)^\wedge \rightarrow A^\wedge$ is continuous with respect to the relative weak* topologies.*

**Proof.** If $\mu_i(f) \rightarrow \mu(f)$ for all $f$ in $L^1(A, G; \alpha)$ then clearly if $\mu(f) \neq 0$ we have $\mu_i(af)/\mu_i(f) \rightarrow \mu(af)/\mu(f)$ for all $a$ in $A$; i.e. $\pi(\mu_i) \rightarrow \pi(\mu)$ is the weak* topology on $A^\wedge$.

**Lemma 4.** *Suppose $\mu_i = m_i \otimes \psi_i \rightarrow \mu = m \otimes \psi$ in $L^1(A, G; \alpha)^\wedge$ with the weak* topology. Then $\psi_i \rightarrow \psi$ in the weak* topology on $L^\infty(G)$.*

**Proof.** Let $a$ be any element of $A$ such that $a_m = 1$ and let $f$ be an arbitrary function in $L^1(G)$. Then

$$\left|\int_G f(g)(\psi_i(g)-\psi(g))\,dg\right| \le \left|a_{m_i}\int_G f\psi_i dg - a_m\int_G f\psi\,dg\right| + |a_m - a_{m_i}|\left|\int_G f\psi_i dg\right|$$

$$\le \left|a_{m_i}\int_G f\psi_i dg - a_m\int_G f\psi\,dg\right| + |a_m - a_{m_i}|\int_G |f|\,dg.$$

Since $\mu_i \to \mu$ in the weak* topology we know that $|\mu_i(a \otimes f) - \mu(a \otimes f)| \to 0$, and this is the first term on the right of the inequality. On the other hand, the second term $|\pi(\mu)(a) - \pi(\mu_i)(a)|\,\|f\|_1 \to 0$ by Lemma 3. Thus $\psi_i \to \psi$ in the weak* (i.e. $L^1(G)$) topology of $L^\infty(G)$.

**Lemma 5.** *$G^\wedge$ acts as an effective topological transformation group on $L^1(A, G; \alpha)^\wedge$.* (See [7].)

**Proof.** The action $(m \otimes \psi_m\chi_1, \chi_2) \to m \otimes \psi_m\chi_1\chi_2$ is obviously effective since distinct characters of $G$ produce distinct functionals on the subalgebras $a \otimes L^1(G)$ as noted in proving Lemma 2. We must show that the action is jointly continuous.

Consider $\mu_i \to \mu$ in $L^1(A, G; \alpha)^\wedge$ and $\chi_j \to \chi$ in $G^\wedge$, where convergence is with respect to the weak* topology in either case. In considering the weak* topology on $L^1(A, G; \alpha)^\wedge$ it suffices to study convergence on functions in $L^1(A, G; \alpha)$ of the form $a \otimes f$ where $a$ is in $A$ and $f$ is a function of compact support in $L^1(G)$. Thus we need only show that in all such cases

$$a_{m_i}\int_G f(g)\psi_{m_i}(g)\chi_j(g)\,dg \to a_m\int_G f(g)\psi_m(g)\chi(g)\,dg.$$

We know, moreover, by Lemmas 3 and 4 that $a_{m_i} \to a_m$ and that $\psi_{m_i} \to \psi_m$ in the weak* topology on $L^\infty(G)$. Thus we need only show that

$$\left|\int_G f(g)\left[\psi_{m_i}(g)\chi_j(g) - \psi_m(g)\chi(g)\right]dg\right| \to 0.$$

We have

$$\left|\int_G f(\psi_{m_i}\chi_j - \psi_m\chi)\,dg\right| \le \left|\int_G f\psi_{m_i}(\chi_j - \chi)\,dg\right| + \left|\int_G f(\psi_{m_i} - \psi_m)\chi\,dg\right|$$

and the second term on the right vanishes by the weak* convergence $\psi_{m_i} \to \psi_m$. However, it is well known (see [15]) that weak* convergence on $G^\wedge$ is equivalent to uniform convergence on compacta and since $f$ is assumed to be of compact support $K$ the first term converges to zero uniformly in $i$:

$$\left|\int_G f\psi_{m_i}(\chi_j - \chi)\,dg\right| \le \sup_{g\in K}|\chi_j(g) - \chi(g)|\,\|f\|_1 \to 0.$$

Consequently the action of $G^\wedge$ on $L^1(A, G; \alpha)^\wedge$ is jointly continuous.

Let $C^1(G)$ be the set of Borel measurable functions $\psi$ on $G$ with values in the unit circle and such that $\psi(e) = 1$. Give $C^1(G)$ the relative weak* topology of

$L^\infty(G)$ (i.e. the $L^1$ topology). Then precisely the same arguments used for the last part of Lemma 5 yield:

**Remark.** $G^\wedge$ acts as an effective topological transformation group on $C^1(G)$.

**Lemma 6.** *Suppose $n \to m$ in $A^\wedge$. Then, for each $n$ there is $\chi_n$ in $G^\wedge$ such that $\psi_n\chi_n \to \psi_m$ in the weak* topology.*

**Proof.** Each member of $A^\wedge$ extends to $A^-$, the double-centralizer algebra of $A$, and $(A^-)^\wedge$ is a compactification of $A^\wedge$. Thus $\alpha_m(g, \gamma) = \psi_m(g)\psi_m(\gamma)/\psi_m(g\gamma)$ is a continuous function of $m$ in $A^\wedge$ and $\alpha_n(g, \gamma) \to \alpha_m(g, \gamma)$ a.e. as $n \to m$ in the weak* topology of $A^\wedge$. By considering $\alpha_n' = \alpha_n/\alpha_m$ and $\psi_n' = \psi_n/\psi_m$, we can reduce the problem to the case where $\alpha_n' \to 1$ and we must show that $\psi_n'\chi_n \to 1$ in the weak* topology.

Since $|\psi(g)| \equiv 1$, all these functions $\psi$ lie in the unit ball of $L^\infty(G)$, which is a compact metric space under the weak* topology. ($G$ is second countable.) Moreover, either $G^\wedge$ is closed in this space or its closure is $G^\wedge \cup \{0\}$. Since $A$ is also second-countable, $A^\wedge$ is a metric space and, without loss of generality, we can assume $n$ to be a sequence, $n_i$. For each $n$, there is a closest point of $G^\wedge \cup \{0\}$ to $\psi_n'$. If this closest point is in $G^\wedge$ call it $\phi_n$, if not, let $\phi_n$ be a member of $G^\wedge$ within $2^{-i}$ of 0 where $n = n_i$.

Now suppose $\psi_n' - \phi_n \not\to 0$. Then there is a subsequence of $n$ such that the distance from $\psi_n'$ to $\phi_n$ always exceeds some fixed $\epsilon > 0$ for all $n$ in the subsequence. Since the unit ball is compact, however, this subsequence has, in turn, a convergent subsequence. Thus if $\psi_n' - \phi_n \not\to 0$ we can find a convergent subsequence of $\psi_n'$ which does not converge to a point of $G^\wedge \cup \{0\}$. We know, however, that $\psi_n'(g)\psi_n'(\gamma) - \psi_n'(g\gamma) \to 0$ a.e., so for any $f$ and $h$ in $L^1(G)$ we have

$$\int_G \int_G f(g)h(\gamma)[\psi_n'(g)\psi_n'(\gamma) - \psi_n'(g\gamma)]\,dg\,d\gamma \to 0$$

(by the Lebesgue bounded convergence theorem). But this can be written as

$$\int_G f(g)\psi_n'(g)\,dg \int_G h(\gamma)\psi_n'(\gamma)\,d\gamma - \int_G f * h(g)\psi_n'(g)\,dg \to 0.$$

This implies that any weak* limit point of the $\psi_n'$ must be a multiplicative functional on $L^1(G)$ and hence in $G^\wedge \cup \{0\}$ which is the set of all such functionals. Consequently we must have $\psi_n' - \phi_n \to 0$. By the remark preceding this lemma, multiplication by $\bar{\phi}_n$ is jointly continuous, so taking $\chi_n = \bar{\phi}_n$ we have $\psi_n'\chi_n \to 1$, which proves the lemma.

**Lemma 7.** *The weak* topology of $A^\wedge$ is the quotient topology induced by $\pi$.*

**Proof.** By Lemma 3, $\pi$ is continuous. We must show that if $\pi^{-1}(S)$ is open then $S$ is open. In particular, suppose $m$ to be in $S$ and $m \otimes \psi_m$ to be in $\pi^{-1}(m)$.

If $\pi^{-1}(S)$ is open there is a weak* neighborhood of $m \otimes \psi_m$ contained in $\pi^{-1}(S)$. Such a neighborhood is defined by a finite family of functions and $\epsilon > 0$; $n \otimes \psi_n\chi$ is in the neighborhood iff

$$\left|\int_G [f_m(g)\psi_m(g) - f_n(g)\psi_n(g)\chi(g)]dg\right| < \epsilon$$

for each $f$ in the finite family. (Note that $n \otimes \psi_n\chi$ is in $\pi^{-1}(S)$ iff $n$ is in $S$.) It suffices to exhibit $\chi_n$ in $G^\wedge$ and, for each $f$ in $L^1(A, G; \alpha)$, a weak* neighborhood of $m$ such that if $n$ is in the neighborhood the above inequality holds with $\chi = \chi_n$. Then if $n$ is in the intersection of such neighborhoods for each of a finite family of $f$'s, $n \otimes \psi_n\chi_n$ is in $\pi^{-1}(S)$ and hence $n$ is in $S$. Thus every $m$ in $S$ will have a neighborhood in $S$.

To this end, we note that

$$\left|\int_G (f_m\psi_m - f_n\psi_n\chi_n)dg\right| \le \left|\int_G (f_m - f_n)\psi_n\chi_n dg\right| + \left|\int_G f_m(\psi_m - \psi_n\chi_n)dg\right|$$
$$\le \int_G |f_m - f_n| dg + \left|\int_G f_m(\psi_m - \psi_n\chi_n)dg\right|.$$

Since $f_n(g) = n(f(g))$ is a continuous function of $n$ on $A^\wedge$, $f_n(g)$ converges to $f_m(g)$ a.e. as $n$ approaches $m$. Moreover $|f_n(g)| \le \|f(g)\|$ and since $f$ is Bochner integrable $\|f\|$ is in $L^1(G)$. By Lebesgue's bounded convergence theorem, therefore, $f_n$ converges to $f_m$ in $L^1(G)$. Thus we can find a weak* neighborhood $N_1$ of $m$ in $A^\wedge$ such that $\int_G |f_m - f_n| dg < \epsilon/2$ for $n$ in $N_1$. On the other hand, Lemma 6 guarantees that the $\chi_n$ can be selected so that $\psi_n\chi_n \to \psi_m$ in the weak* topology, so we can find a weak* neighborhood $N_2$ of $m$ such that $|\int_G f_m(\psi_m - \psi_n\chi_n)\, dg| < \epsilon/2$ for $n$ in $N_2$. Thus if $n$ is in $N_1 \cap N_2$ we conclude that

$$\left|\int_G (f_m\psi_m - f_n\psi_n\chi_n)dg\right| < \epsilon.$$

Consequently $S$ is open whenever $\pi^{-1}(S)$ is open.

Let $C^1(G)$ denote, as before, the set of Borel measurable functions $\psi$ on $G$ with values in the unit circle such that $\psi(e) = 1$, and let $S^2(G)$ be the set of symmetric cocycles $\alpha$ on $G \times G$. Both sets are groups and with the topology of local convergence in measure (i.e. convergence in measure on compact sets) they are topological groups. Moreover, the topology induced on the subgroup $G^\wedge$ of $C^1(G)$ is equivalent to the weak* topology on $G^\wedge$ and the coboundary map $\partial\colon C^1 \to S^2$, defined by $(\partial\psi)(g, \gamma) = \psi(g)\psi(\gamma)/\psi(g\gamma)$, is continuous. We will have a short exact sequence of topological groups $1 \to G^\wedge \to C^1(G) \xrightarrow{\partial} S^2(G) \to 1$ provided we can show that the topology on $S^2(G)$ is the quotient topology. This requires a stronger version of Lemma 6.

**Lemma 8.** *Suppose $\alpha_n \to \alpha$ locally in measure (or a.e.) on $G \times G$, where $\alpha_n = \partial\psi_n$, $\alpha = \partial\psi$. Then for each $n$ there is $\chi_n$ in $G^\wedge$ such that $\psi_n\chi_n \to \psi$ locally in measure (or a.e.) on $G$.*

**Proof.** Just as in Lemma 6 we can reduce the problem to the case where $\psi \equiv 1$. Moreover, the proof of Lemma 6 with minor alterations shows that we have $\psi_n \chi_n \to 1$ in the weak* topology on $C^1(G)$.

Since $\alpha_n \to 1$, we know $\psi_n(g)\psi_n(\gamma) - \psi_n(g\gamma) \to 0$ locally in measure (or a.e.) on $G \times G$. Let $E$ be a compact neighborhood of measure $\mu(E)$ in $G$. Then

$$\frac{1}{\mu(E)} \int_E [\psi_n(g)\psi_n(\gamma) - \psi_n(g\gamma)]\chi_n(g\gamma)\,dg \to 0$$

locally in measure (or a.e.) on $G$. But this implies

$$\frac{1}{\mu(E)}\left[\psi_n(\gamma)\chi_n(\gamma) \int_E \psi_n(g)\chi_n(g)\,dg - \int_E \psi_n(g\gamma)\chi_n(g\gamma)\,dg\right] \to 0$$

locally in measure (or a.e.) on $G$. But since $\psi_n \chi_n \to 1$ in the weak* topology, both the integrals in the last expression converge to $\mu(E)$ and it follows that $\psi_n \chi_n \to 1$ locally in measure (or a.e.).

**Lemma 9.** *The topology of $S^2$ is the quotient topology induced by $\partial$.*

**Proof.** We know $\partial$ is continuous so, as in Lemma 7, the crucial question is the existence of the "cross-section", provided here by Lemma 8. We must show that if $\partial^{-1}(S)$ is open then $S$ is open. However, if $\alpha_n \to \alpha$ with $\alpha_n$ in the complement of $S$ we would have $\psi_n \chi_n \to \psi$ with $\partial \psi_n = \alpha_n$ and $\partial \psi = \alpha$ by Lemma 8. Since $G^\wedge$ is the kernel of $\partial$, it follows that $\psi_n \chi_n$ is in the complement of $\partial^{-1}(S)$. Thus $\partial^{-1}(S)$ is not open if $S$ is not.

We have now proved that $1 \to G^\wedge \to C^1(G) \to S^2(G) \to 1$ is a short exact sequence of topological groups. In particular $C^1$ is a $G^\wedge$-principal bundle over $S^2$.

**Theorem 1.** *$L^1(A, G; \alpha)^\wedge$ is a principal fibre bundle with fibre $G^\wedge$ and base space $A^\wedge$. It is, in fact, the bundle over $A^\wedge$ induced from $C^1(G)$ by the mapping, $m \to \alpha_m$, of $A^\wedge$ into $S^2(G)$ produced by $\alpha$.*

**Proof.** We use the definition of principal bundle and induced bundle given in [7]. By Lemma 5, $L^1(A, G; \alpha)^\wedge$ is an effective $G^\wedge$-space. By Lemma 7 the quotient topology on $A^\wedge$ is the usual topology and continuity of translation follows from Lemma 8. It is evident that the bundle is induced by the mapping produced by $\alpha$.

The theorem has an alternative interpretation in terms of the homogeneous Banach*-algebraic bundles discussed by Fell [6]. Busby has shown [2] that in the second-countable case every twisted group algebra is the $L^1$ cross-section algebra of such a bundle and vice-versa. Thus we have

**Corollary.** *Every second-countable commutative homogeneous Banach*-algebraic bundle over $G$ has as dual a principal fibre bundle with fibre $G^\wedge$ and base* *space $A^\wedge$, where $A$ is the fibre over the identity in the Banach*-algebraic bundle. ($G$ and $A$ must be commutative if the bundle is.)*

It is interesting to note that since the group algebra of a group extension (all groups locally compact and second countable) is a twisted group algebra (see [4]) we could also get the classical duality results for Abelian group extensions from the theorem.

4. **Twisted group algebras of principal bundles.** We consider next the converse problem: given an appropriate principal bundle, is there a twisted group algebra (or equivalently, homogeneous $B^*$ algebraic bundle) to which it is dual? For this purpose we define a class of bundles which we will call characteristic principal bundles. We retain some of our previous notation and conventions rather than starting anew. For instance, $G$ is to be a second countable locally compact Abelian group, etc.

Although $\alpha_m$ is trivial for each fixed $m$, $\alpha_m(g, \gamma) = \psi_m(g)\psi_m(\gamma)/\psi_m(g\gamma)$, $\alpha$ need not be trivial because it may not be possible to choose $\psi$ to be a unitary double-centralizer of $A$. For instance, if $A$ is $C(X)$, the continuous functions on a compact Hausdorff space $X$, the unitary double-centralizers of $A$ are the continuous functions on $X$ taking values in the unit circle. It may be that $\alpha_m$ is a continuous function of $m$ in $A^\wedge = X$ while it is not possible to choose $\psi_m$ continuous. This situation is illustrated in the following example. Let $G = G^\wedge = Z_2$ and let $X = A^\wedge$ be a circle. We must have $\alpha(0, 0) = \alpha(0, 1) = \alpha(1, 0) = 1$. However $\alpha(1, 1)$ can be chosen as any map of the circle $X$ to the unit circle. We can easily compute $\psi(0) = 1$ and $(\psi(1))^2 = \alpha(1, 1)$. Thus $\psi$ is continuous—and $\alpha$ is trivial—if and only if the Brouwer degree of the mapping $\alpha(1, 1)$ is even. (Note that all possible values of $\psi_m$ differ by a character of $Z_2$.) Two distinct bundles appear as $L^1(A, G; \alpha)^\wedge$. If $\alpha(1, 1)$ has even degree, the bundle is trivial: $X \times Z_2$. If $\alpha(1, 1)$ has odd degree we have the nontrivial $Z_2$ principal bundle over the circle $X$. We shall find this reflects the fact that there are exactly two isomorphism classes of $A$-valued commutative group algebras over $G$ corresponding to these bundles.

Let $X$ be a principal $G^\wedge$ bundle over $M$ (i.e. $G^\wedge$ acts effectively as a topological transformation group on $X$ with continuous translation and $M = X/G^\wedge$, the orbit space). Suppose $M$ is a second countable, locally compact Hausdorff space. We say $X$ is *characteristic* if there exists a function $F$ on $G \times X$ to the unit circle, measurable on $G$, continuous on $X$, with $F(e, t) = 1$ and $F(g, \chi \cdot t) = \chi(g)F(g, t)$ for all $g$ in $G$, $\chi$ in $G^\wedge$ and $t$ in $X$. It is easy to prove that any $G^\wedge$ space for which such a function $F$ exists must be a principal $G^\wedge$ bundle. (A function $f$ on a $G^\wedge$ space $X$ having the property $f(\chi t) = \chi(g)f(t)$ has been called an eigenfunction with eigenvalue $g$ by Keynes and Robertson in [10] where such functions are used to study ergodicity and mixing properties.) For such an $X$ we have:

**Lemma 10.** *$X = L^1(A, G; \alpha)^\wedge$, where $A = C_\infty(M)$ the continuous functions vanishing at $\infty$ on $M = X/G^\wedge$ and $\alpha(g, \gamma)$ is the function defined at $m = G^\wedge t$ by $\alpha_m(g, \gamma) = F(g, t)F(\gamma, t)/F(g\gamma, t)$.*

**Proof.** We note that $\alpha_m$ is well defined, since $F(g, \chi \cdot t) = \chi(g)F(g, t)$. By the previous theorem, $L^1(A, G; \alpha)^\wedge$ consists of the pairs $(m, F(-, t) \cdot \chi) = (m, F(-, \chi \cdot t))$ in $M \times C^1(G)$, where $t$ is a particular representative of the orbit $m$. Since $G^\wedge$ acts effectively on $X$ there is a one-to-one correspondence of $\chi t$ to $(m, F(-, t)\chi)$ between $X$ and $L^1(A, G; \alpha)^\wedge$. On the other hand, $M$ has the quotient topology induced by $\pi: X \to X/G^\wedge = M$ which is also the weak $A$ topology of $M = A^\wedge$ and $F(g, \xi)$ is continuous in $\xi$. Consequently, if $\mu_i \to \mu$ in $X$, $\pi(\mu_i) \to \pi(\mu)$ in $M$ and $F(-, \mu_i) \to F(-, \mu)$ in $C^1(G)$ while, conversely, if $m_i \to m$ and $F(-, t_i)\chi_i \to F(-, t)\chi$ a.e. then $t_i\chi_i \to t\chi$ in $X$.

**Lemma 11.** *If $F_1$ and $F_2$ are two functions on $X$ having the properties described above then $F_1/F_2$ defines a continuous function on $M$.*

**Proof.** The function $F_1/F_2$ is clearly constant on orbits $G^\wedge t$, since $F_1(g, \chi t)/F_2(g, \chi t) = \chi(g)F_1(g, t)/\chi(g)F_2(g, t)$. Since $F_1$ and $F_2$ are continuous on $X$, $F_1/F_2$ is also, and since $F_1/F_2$ is constant on orbits it defines a continuous function on $M = X/G^\wedge$, which carries the quotient topology.

Because different commutative Banach algebras can have the same maximal ideal spaces, we can expect no general 1-1 correspondence. However Busby and the author have shown [4] that for purposes of representation theory (and hence in studying maximal ideal spaces) the algebra $A$ in a twisted group algebra can be replaced by its enveloping $C^*$-algebra. The following is thus the strongest possible theorem in our context.

**Theorem 2.** *Let $A$ be a separable commutative $C^*$-algebra and $G$ a second-countable locally compact Abelian group. Then there is a 1-1 correspondence between the isomorphism classes of commutative twisted group algebras $L^1(A, G; \alpha)$ and the characteristic principal $G^\wedge$ bundles over $A^\wedge$.*

**Proof.** The correspondence in one direction follows from the results of the previous section. Starting with $L^1(A, G; \alpha)$ one can construct $F$ and retrieve $\alpha$ as in Lemma 10. On the other hand, by Lemma 11 any two $\alpha$'s constructed in this way must be cohomologous and, by Theorem 2.7 of [4], produce isomorphic twisted group algebras.

We have been using the definition of principal bundle introduced in [7], which does not require local triviality as does the classical definition of [16]. J. M. G. Fell has remarked that not even every locally trivial principal $G^\wedge$ bundle is characteristic, for, by the Borsuk-Ulam theorem there is no antipodes preserving map of $S^2$ to $S^1$. Thus $S^2$ is a locally trivial principal $Z_2$ bundle over $RP^2$ but not characteristic. On the other hand every second countable locally compact Abelian group extension is a characteristic principal bundle, but a countable product $Z_2 \to T \to T/Z_2$, with $T$ the unit circle, is not locally trivial.

Suppose $G$ is compact. For any $r$ in $X$, define $Q_r$ on $G^\wedge \times X$ to be the Fourier transform

$$Q_r(\chi, t) = \int_G \chi(g)F(g, t)/F(g, r)\,dg.$$

**Lemma 12.** *For $G$ compact, $X$ is locally trivial if and only if, for each $r$ in $X$, $F$ can be chosen so that $Q_r(\chi, t) \to Q_r(\chi, r)$ uniformly in $\chi$ as $t \to r$.*

**Proof.** If $X$ is locally trivial we can construct $F$ having the desired property quite easily.

Suppose, conversely, that $Q_r$ has the property. We know that $Q_r(1, r) = 1$, $Q_r(\chi, r) = 0$ if $\chi \neq 1$. By the uniformity assumption, we can find an open neighborhood $U$ of $r$ such that, for all $t$ in $U$, $Q_r(1, t) > ½$ and $Q_r(\chi, t) < ½$ when $\chi \neq 1$. Since $Q_r(\chi, t) = Q_r(1, \chi t)$, $\chi U \cap U$ is empty for $\chi \neq 1$. Thus $U$ is a local cross-section over the open set $\pi(U)$.

**Remark.** If $G$ is finite then every characteristic principal $G^\wedge$ bundle is locally trivial.

**Proof.** Since $G^\wedge = G$ is finite, we have the uniformity required in Lemma 12.

We have demonstrated a 1-1 correspondence between the homeomorphism classes of characteristic principal $G^\wedge$-bundles over $M$ and the twisted group algebras defined on $G$ with values in $C_\infty(M)$. By results in [4] and [2] these also correspond to the symmetric cohomology groups on $G$ with values in the circle valued functions on the Stone-Čech compactification of $M$ and to the isomorphism classes of homogeneous $B^*$-algebraic bundles with $C_\infty(M)$ the fibre over the identity.

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DEPARTMENT OF MATHEMATICS, OAKLAND UNIVERSITY, ROCHESTER, MICHIGAN 48063